Bose-Einstein condensation in a system of q-bosons

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We introduce a q-boson system with infinite range interaction which exhibits in the thermodynamic limit a Bose-Einstein condensation. Young tableaux analysis is used to solve the quantum problem of the system and to show that for a particular value of the deformation parameter the system behaves as a Bose gas with hard-core repulsion on a full graph. The method used to solve our system is of general validity and can be applied to other Hamiltonians with infinite range interactions.

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The possibility that a system of interacting bosons can undergo a Bose-Einstein condensation represents one of most fascinating problems in the area of statistical physics. This possibility was first conjectured by London in connection with the λ transition of the liquid ^{4}He and since then much work has been done on this problem [1]. In spite of this, a rigorous proof of the occurrence of a Bose condensation in a system of interacting bosons is still lacking. Very recently, Tòth and Penrose considered this problem in connection with a Bose gas with hard-core repulsion on a complete graph [2,3]. These authors calculated, by different methods, the thermodynamic free energy per site and showed the existence in the system of a Bose-Einstein condensation. The aim of the present paper is to consider the above problem in connection with a system of q-bosons. More precisely, we introduce a system of q-bosons on a full graph which is exactly solvable through the number state method (NSM) and which, for a particular value of the deformation parameter, reduces to the Bose gas with hard-core repulsion considered by Tòth and Penrose. This proves the existence, in our system, of a Bose-Einstein condensation. Besides the hard-core repulsion, our model includes other type of interactions, such as the ones which put no more than two particles per site, etc., and it allows us to deform continuously from one type to another. We think the interest of this paper is twofold: it shows the existence of a Bose-Einstein condensation in a system of q-bosons and it provides an example in which a q-deformed algebra (quantum group) is effectively used to solve a concrete physical problem. Furthermore we remark that the method used to solve our system is of general validity and can be applied to other Hamiltonians which are invariant under the permutation group.

Let us start by introducing the following Hamiltonian:

$$H = -\frac{1}{f}\sum_{i,j}^{f} m_{ij} b_i^\dagger b_j + N, \tag{1}$$

where $b^\dagger$ and b are q-boson creation and annihilation operators satisfying the deformed Heisenberg algebra

$$[b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0, \quad [b_i, b_j^\dagger] = \left(1 + q\, b_j^\dagger b_i\right)\delta_{i,j} \tag{2}$$

and N is the corresponding q-deformed number operator [4,5]. This deformed algebra (also called quantum group H_q) implies the following action of the creation and annihilation operators on the Fock space of states of a single particle [4]:

$$b_i|n\rangle = \sqrt{[n]_q}\,|n-1\rangle \tag{3}$$

$$b_i^\dagger|n\rangle = \sqrt{[n+1]_q}\,|n+1\rangle \tag{4}$$

with $[n]_q = \frac{(1+q)^n - 1}{q}$. When the coupling matrix m_{ij} is taken to be the tridiagonal next neighbor coupling matrix, Hamiltonian (1) represents a quantum chain of anharmonic oscillators coupled through dispersive interactions, also known as the quantum Ablowitz-Ladik system [6]. In this case the system is exactly solvable both by the algebraic Bethe Ansatz [7] and by the NSM [5,4]. This system, being one dimensional, does not possess phase transitions at finite temperatures. On the other hand, it is of interest to consider the case of symmetric couplings, i.e., $m_{ij} = 1 - \delta_{ij}$, for which the Hamiltonian becomes invariant under the action of the group S_f (permutation group of f objects). From a physical point of view this system can be seen as a gas of interacting q-bosons on a lattice which is a complete graph. Since each q-boson interacts equally with each other we have that the system is in a sense "infinite-dimensional" and can exhibit a phase transition at finite temperatures. As it was shown in Ref. [8], a Hamiltonian of type (1) can be block diagonalized by using the conservation of the number operator and by taking advantage of its invariance under the action of the symmetric group S_f. In this paper we will restrict to the case $q = -2$. This case is quite important since it corresponds to a repulsive hard-core interaction between the q-bosons (i.e., a generic site of the lattice can be occupied at most by one particle). For $q = -2$, we have, from Eqs. (2), that the commutation relations become a mixture of bosonic and fermionic ones

$$\{b_i, b_i^\dagger\} = 0,\ [b_i, b_j\dagger] = 0,\ [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0, \tag{5}$$

and the action of the creation and annihilation operators in Eq. (4) on the states $|0\rangle, |1\rangle$ is the same as for fermionic operators. Furthermore, in this limit the q-

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deformed number operator is equivalent to

$$N = \sum_{i=1}^{f} b_i^\dagger b_i \tag{6}$$

so that the Hamiltonian (1) can be rewritten as

$$H = -\frac{1}{f} \sum_{i,j=1}^{f} b_i^\dagger b_j + \sum_{i=1}^{f} b_i^\dagger b_i . \tag{7}$$

In the following we show that systems (7) and (5) have the same thermodynamic properties of the Bose gas with hard-core repulsion considered by Tòth and Penrose. To this end we construct the partition function and the thermodynamic free energy for our system (for $q = -2$) and prove that they coincide with the expressions of Refs. [2,3] for a Bose gas with hard-core repulsion. In order to diagonalize Hamiltonian (7) for arbitrary f (number of lattice sites) and for arbitrary n (number of quanta) we use the conservation of the number operator together with the invariance of H under S_f. More precisely, the fact that $[H, N] = 0$ allows us to decompose the Hilbert space K of quantum states into the direct sum of eigenspaces K_n, corresponding to a fixed eigenvalue n of the number operator. The dimension of these spaces is just the number of ways n quanta (fermions) can be placed on f sites, i.e.,

$$d_n \equiv \dim(K_n) = \frac{f!}{n!(f-n)!} .$$

This leads to the diagonalization (for each n) of a finite $d_n \times d_n$ matrix. On the other hand, the invariance of the Hamiltonian under the action of the permutation group S_f implies that each $d_n \times d_n$ matrix can be further block diagonalized according to the irreducible representations of this group. In Ref. [8] it was shown how to construct the eigenspaces of N with a given S_f symmetry property by using Young tableaux filled with n quanta. Here we recall (for details see Ref. [8]) the rule used to fill in the tableaux with n quanta for a system of q-bosons. First, partition n in the numbers $n_1, n_2, ..., n_k$ such that $\sum_{i=1}^{k} n_i = n$ and then put these numbers in the boxes of a given Young tableau in such a way that they must not increase when moving from left to right in each row, and they must decrease when moving down each column of a given tableau.

The eigenstates of N spanning each irreducible representation of S_f are then constructed by applying symmetrizer and antisymmetrizer operators to each filled tableau. More precisely, by denoting with R a permutation within the rows, by denoting with C a permutation within the columns, and by denoting with δ_C the parity of this permutation, one constructs the operator $\sum_R R \sum_C \delta_C C$ and applies it to each filled tableau (this amounts to symmetrizing each filled tableau with respect to the rows and antisymmetrizing each filled tableau with respect to the columns). One can prove that the dimensions of the blocks into which the Hamiltonian matrix decomposes for each symmetry class of S_f and for a fixed value of n is obtained by counting the different ways in which a given tableau can be filled with n quanta according to the above rule. Furthermore, the number of times a given diagonal block is repeated in H (i.e., the degeneracy of the eigenvalues of that block) is simply the dimension of the corresponding tableau (i.e., the dimension of the corresponding irreducible representation of S_f) [9]. In the case $q = -2$ (i.e., for hard-core repulsion) we have that the anticommutator in Eq. (5) limits the number of quanta which can occupy a generic site of the lattice to be 0 or 1. As a consequence we have that the possible tableaux filled with n quanta cannot have more than two rows (due to the antisymmetry property of a tableau along columns). More precisely, for a fixed value of f and n we can have all the tableaux of type $\{f-r, r\}$ where $r = 0, 1, ..., \min(n, f-n)$. The dimension d_r of these tableaux is obtained by counting the number of different ways they can be filled with the numbers $1, 2, ..., f$ in the standard manner (see [9]). One can easily prove that this number is given by

$$d_r = \frac{f!(f-2r+1)}{r!(f-r+1)!} . \tag{8}$$

Furthermore, for each tableau of a given type there is only one way to fill it with n quanta according to the above rule. This means that in the block diagonalization of H the blocks associated with the possible Young tableaux have dimensions of one. The corresponding eigenvalues are then obtained from the expectation values of H with respect to the states constructed from the corresponding filled tableaux, with the degeneracy of these eigenvalues simply given by Eq. (8). As an illustrative example, let us consider the case $f = 4$, $n = 2$. In this case we can have only the following filled tableaux of type:
$\{f-r, r\}$, $r = 0, 1, 2$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 0 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 0 \\ \hline \end{array} .$$

To each of these tableaux we construct the corresponding states $|\psi_r\rangle$ as

$$|\psi_0\rangle = \frac{1}{\sqrt{(6)}} (|1100\rangle + |0011\rangle + |0101\rangle + |1001\rangle + |1010\rangle + |0110\rangle) ,$$

$$|\psi_1\rangle = \frac{1}{\sqrt{(4)}} (|1100\rangle + |1010\rangle - |0101\rangle - |0011\rangle) ,$$

$$|\psi_2\rangle = \frac{1}{\sqrt{(4)}} (|1100\rangle + |0011\rangle - |0110\rangle - |1001\rangle) ,$$

from which the eigenvalues E_r are obtained

$$E_r = \langle \psi_r | H | \psi_r \rangle \; r = 0, 1, 2. \tag{9}$$

For the above example, we find $E_0 = \frac{1}{2}$, $E_1 = \frac{3}{2}$, $E_2 = 2$

with degeneracy respectively given by $1, 3, 2$. The advantage of this method of diagonalizing H in the subspace of the Hilbert space with n fixed and according to the irreducible representations of S_f, is that it is completely algebraic so that it can be easily implemented on a computer using symbolic languages such as Mathematica. This allows us to compute in a very effective way the eigenvalues of H for arbitrary values of f and n. By fixing n and solving the problem for arbitrary f one can extract recursion relations which allow us to get the general formula for the eigenvalues of H for arbitrary n. Thus, for example, for r=0,1,2,3,4 and arbitrary f and n, we find that to the tableau of type $\{f-r, r\}$ are associated, respectively, the following eigenvalues:

$$E_0 = \frac{n\,(n-1)}{f},$$

$$E_1 = 1 + \frac{n\,(n-1)}{f},$$

$$E_2 = 2 + \frac{n\,(n-1)-2}{f},$$

$$E_3 = 3 + \frac{n\,(n-1)-6}{f},$$

$$E_4 = 4 + \frac{n\,(n-1)-12}{f}.$$

From these expressions we easily get the general result

$$E_r = r + \frac{n\,(n-1) - r\,(r-1)}{f}, \qquad r = 0, 1, ..., \min(f-n, n) \quad (10)$$

with degeneracy given by Eq. (8). From Eqs. (8) and (10) the partition function for our system is readily written as

$$Z(n, f) = \mathrm{tr}(e^{-\beta H}) = \sum_{r=0}^{\min(f-n,n)} d_r e^{-\beta E_r}, \qquad (11)$$

where β denotes the inverse temperature. We think it is remarkable that this expression coincides with the partition function of the Bose gas with hard-core repulsion on a complete graph considered in Refs. [2,3], this proving the thermodynamic equivalence of the two systems. We remark that in our approach the hard core repulsion has been included through a suitable deformation of the commutation relations, this providing an alternative way of dealing with quantum systems of interacting particles. The fact that our system, for $q = -2$, has the same partition function as the system studied by Tòth and Penrose implies that it exhibits a phase transition for $\beta = \beta^*(\rho)$, where

$$\beta^*(\rho) = \frac{1}{1-2\rho} \ln \frac{1-\rho}{\rho} \qquad (12)$$

and ρ is the number of particle per site ($\rho \in [0, 1]$). Furthermore, one can prove that this phase transition is actually a Bose-Einstein condensation, with the condensate density given by $\rho - \rho^*(\beta)[1 - \rho - \rho^*(\beta)]$ if $\beta \geq \beta^*(\rho)$ and equal to 0 if $\beta \leq \beta^*(\rho)$. This directly follows from Sec. (9) of Ref. [3] after noting that the operator giving the number of particles in the zero state in our case coincides with operator A^+A of Ref. [3]. This proves the existence in our system of a Bose-Einstein condensation.

We remark that the method of solution presented here works for arbitrary values of q. In these cases, however, one finds the blocks in the decomposition of H to depend both on f and n, thus making the solubility of the model more involved. In the case in which the deformation parameter is a k-th root of unity (i.e., the system cannot have more than k q-bosons per site), the block diagonalization of H involves only Young tableaux with no more than $k+1$ rows, this simplifying the problem. Finally, we remark that the method used to solve this q-bose system can be generalized to fermionic systems invariant under the permutation group such as the Hubbard model with infinite range hopping [10].

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